



## May-Wigner Instability Scenario :

"Will a Large Complex System be Stable?"

This question was posed by **Robert May** (*NATURE* 238, 413 (1972)) who introduced a toy **linear** model for (in)stability of a large system of many interacting species:

$$\dot{\mathbf{x}} = -\lambda \mathbf{x} + B\mathbf{x}; \quad \lambda > 0; \mathbf{x} \in \mathbb{R}^N$$

Without interactions the part  $\dot{\mathbf{x}} = -\lambda \mathbf{x}$  describes a **simple exponential** relaxation of  $N$  uncoupled degrees of freedom  $x_i$  with the same rate  $\lambda > 0$  towards the **stable equilibrium**  $\mathbf{x} = 0$ . A complicated interaction between dynamics of different degrees of freedom is mimicked by a general **real asymmetric**  $N \times N$  **random matrix**  $B$  with mean zero and prescribed variance  $\rho^2$  of all entries. As a typical eigenvalue of  $B$  with the largest real part grows as  $\rho \sqrt{N}$  the equilibrium  $\mathbf{x} = 0$  becomes **unstable** as long as  $\rho < \lambda \sqrt{N}$ .

This scenario is known in the literature as the "**May-Wigner** instability" and despite its oversimplifying and schematic nature attracted very considerable attention in mathematical ecology and complex systems theory over the years.



## Counting equilibria via Kac-Rice formulae:

A standard analysis of autonomous ODE's starts with finding **equilibrium points** and classifying them by **stability** properties.

We would like to know the **total number**  $N_{tot}(D)$  of all possible **equilibria** of our system of nonlinear ODEs, i.e. the number of simultaneous solutions of  $N$  equations

$$x_j + f_j(x_1, \dots, x_N)$$

## Mean number of equilibria and the Elliptic Ensemble:

Using Kac-Rice approach we are able to count the **mean** total number  $E f N_{tot} g$  of all possible **equilibria** in the system of nonlinear ODEs under consideration. This turns out to be given by (**YVF & Khoruzhenko**, *in progress*):

$$E f N_{tot} g = \frac{1}{m^N N^{(N-1)/2}} \int_0^D \det (m + t^{\rho-})^{\rho} \frac{1}{N} \int_{\mathbf{X}} e^{-\frac{Nt^2}{2}} dt$$

where  $m = \frac{1}{c}$  with some characteristic scale  $c$ , and the random **real asymmetric** matrix  $\mathbf{X}$  being taken from the **Gaussian Elliptic Ensemble**:

$$P(\mathbf{X}) = C_N(\beta) e^{-\frac{1}{2(1-\beta^2)} [\text{Tr} \mathbf{X} \mathbf{X}^T + \text{Tr} \mathbf{X}^2]}, \quad \beta \in [0; 1]$$

The parameter  $\beta$  depends on the ratio of variances of **gradient** and **solenoidal** components of the field such that the **real Ginibre ensemble** with  $\beta = 0$  corresponds to **purely solenoidal**, and GOE with  $\beta = 1$  to **purely gradient** flow.

Let us denote  $\rho_N^{(r)}(\lambda)$  the mean density of **real** eigenvalues of  $N \times N$  matrices  $\mathbf{X}$  for the elliptic ensemble at  $\beta$ . Then it turns out that (cf. **Edelman, Kostlan, Schub** '94.)

$$h_j \det(\mathbf{X})^j \int_{\mathbf{X}} = 2^{\rho} \frac{(N-1)!}{(N-2)!!} \rho_{N+1}^{(r)}(\lambda) e^{-\frac{\lambda^2}{2}}$$

## A Nonlinear Analogue of May-Wigner Instability as Topology Detrivialization:

The mean density  $\frac{\langle r \rangle}{N}$  of real eigenvalues for the elliptic ensemble was computed explicitly by **Forrester & Nagao '08** in terms of Hermite polynomials, and its large- $N$  asymptotic behaviour was studied as well.

Asymptotic analysis of the counting problem for  $N \gg 1$  reveals then a **topology detrivialization** transition, with the total number of equilibria **abruptly** changing from **a single equilibrium** for  $\beta > c = \frac{1}{N(F''(0) + \rho''(0))}$  to **exponentially many** equilibria as long as  $\beta < c$ :

$$E f N_{tot} \sim \frac{2(1+\beta)}{1} e^{N \cdot tot(m)}; \quad tot(m) = \frac{m^2 - 1}{2} \quad \ln m > 0 \quad \text{for } m = \frac{1}{\beta} < 1$$

## Landscape topology (de)trivialization for gradient dynamics:

In the case of purely **gradient** dynamics  $\dot{\mathbf{x}} = -\nabla V(\mathbf{x}) = -\nabla L(\mathbf{x})$  where :

$$L(\mathbf{x}) = \frac{\rho}{2} \sum_{i=1}^N x_i^2 + V(x_1, \dots, x_N); \quad \rho > 0; \quad -1 < x_i < 1$$

is the Lyapunov function (or "energy functional"). Correspondingly the equilibria points are simply **stationary** points of the Lyapunov function whereas the stable equilibria are local **minima**.

Taking as before  $V(\mathbf{x})$  to be **stationary isotropic** random Gaussian field with covariance structure  $E[V(\mathbf{x})V(\mathbf{y})] = F(\mathbf{x}, \mathbf{y})$

## Landscape topology (de)trivialization for gradient dynamics:

The asymptotics  $F_{N-1}(t)$  is well known (Tracy & Widom '94; Borot et al '11).  
Using it for a fixed  $m \notin 1$  we find for the mean number of minima:

$$\mathbb{E} fN_{min} \sim \begin{cases} 1; & m > 1 \\ e^{N_{st}(m)}; & m < 1 \end{cases}$$

Here the complexity of stable equilibria (minima) is given by

$$N_{st}(m) = \frac{1}{2}(m^2 - m)$$



## Part II: Topology of Random Algebraic Varieties :

Recently, the problem of computing the expectation of topological properties of random algebraic varieties has attracted a lot of interest ( see e.g. the works by **Burgisser '07, Nazarov-Sodin '09, Gayet-Welshinger '11, Sarnak '11, Lerario-Lundberg '12, Sarnak-Wigman '13**) and others. An important class of problems addresses estimates for Betti numbers of "generic" (=random) real hypersurfaces given by **zero set** of real random homogenous polynomials of degree  $d$  in  $n + 1$  variables restricted to the unit sphere. E.g. for  $d = 60$  and  $n = 2$  a typical picture is:

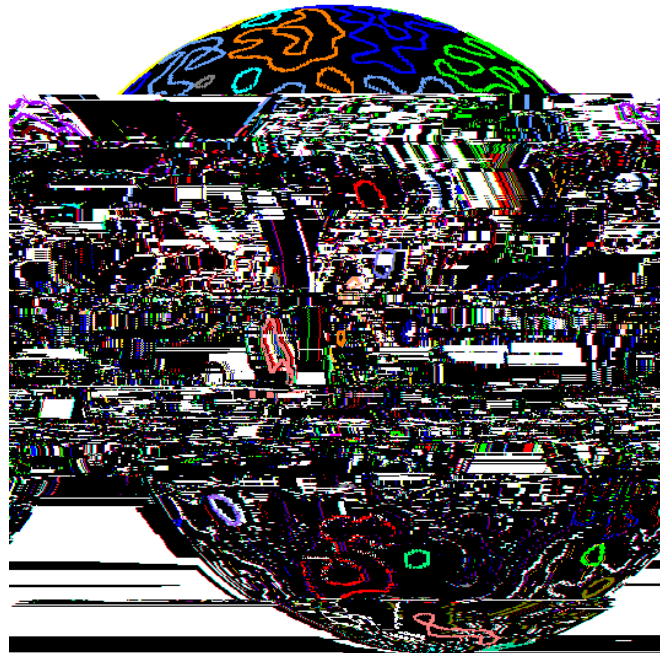


Figure 1: Zero locus of a random polynomial of degree  $d = 60$  on the sphere ( **M. Nastasescu**)

## Upper bound on $b_0$ by Random Matrix Theory:

It turns out that the methods and results just exposed allow one to provide a useful **upper bound** to the **expected number of connected components**  $b_0(f)$ . Indeed, every component of the zero locus of the polynomials restricted to the sphere bounds a region where the function attains at least a maximum or a minimum, and consequently  $E f b_0(f) g = E f N_{min} + N_{max} g$ , where  $N_{min=max}$  are numbers of minima/maxima on the sphere. The problem then amounts to counting minima of a random function on a sphere.

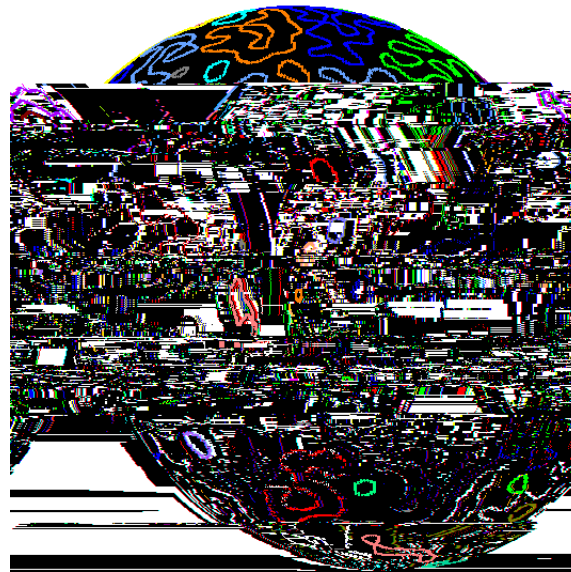


Figure 2: Zero locus of a random polynomial of degree  $d = 60$  on the sphere ( [M. Nastasescu](#) )

## Counting Stationary points for Isotropic Gaussian Landscapes:

In recent years there was a steady progress in counting & classifying the **mean number** of **stationary points** of smooth isotropic Gaussian random fields  $V(\mathbf{x})$  on the sphere  $|\mathbf{x}| = R$  such that

$$E \int_{|\mathbf{x}|=R} \nabla V(\mathbf{x}) \cdot \nabla V(\mathbf{x}^\theta) g = F(\mathbf{x}, \mathbf{x}^\theta)$$

Using the multidimensional **Kac-Rice** integrals it was shown, in particular, that  $E \int_{|\mathbf{x}|=R} \nabla V(\mathbf{x}) \cdot \nabla V(\mathbf{x}^\theta) g$  can be again directly related to the the distribution  $F_N(t)$  of the maximal eigenvalue of **random GOE** matrices  $H$  such that  $P(H) \propto \exp\left(-\frac{N}{4} \text{Tr} H^2\right)$ . Namely

$$E \int_{|\mathbf{x}|=R} \nabla V(\mathbf{x}) \cdot \nabla V(\mathbf{x}^\theta) g = 2 \frac{1+B}{1-B} \int_0^{\infty} \rho \frac{1}{B} R \frac{1}{1} e^{-NBt^2}$$

## Upper bound on $b_0$ for Gaussian rotationally invariant polynomials:

Endowing polynomials with a rotationally-invariant Gaussian distribution we can find  $E f N_{min} g$  for any  $n$  and  $d$  from our formalism. We will mostly be interested in the limits  $d \rightarrow \infty$  for a fixed  $n$  or  $n \rightarrow \infty$  for a fixed  $d$ .

Let  $f Y_l^j g$  denote the standard basis of **spherical harmonics** of degree  $l$  on sphere  $S^n$ , then a random invariant Gaussian polynomial of degree  $d$  in  $n + 1$  variables can be constructed as :

$$f(x) = \prod_{d \in \mathbb{N}} p_d(l) \prod_j \frac{j}{|j|} x_j^{d-l} Y_l^j \left( \frac{x}{|x|} \right) ; p_d(l) \geq 0 \quad \text{Kostlan}$$

where  $\frac{j}{|j|}$  are i.i.d. Gaussian coefficients, and nonnegative weights  $p_d(d); p_d(d-2); \dots$  parametrize *all* invariant ensembles.

We assume that there exists such  $0 < \alpha < 1$  that as  $d \rightarrow \infty$  the polynomials assume the scaling form:  $p_d(d-x) d^{-\alpha} (x) \in (\mathbb{Q})^{\mathcal{V}}$  (102)

Summary :

For the special case of **purely gradient** flows one can also find explicit expression for the number of **stable** equilibria. The latter are exponential in  $N$  but their fraction among all equilibria is negligible. The crossover expression in that case is given in terms of the