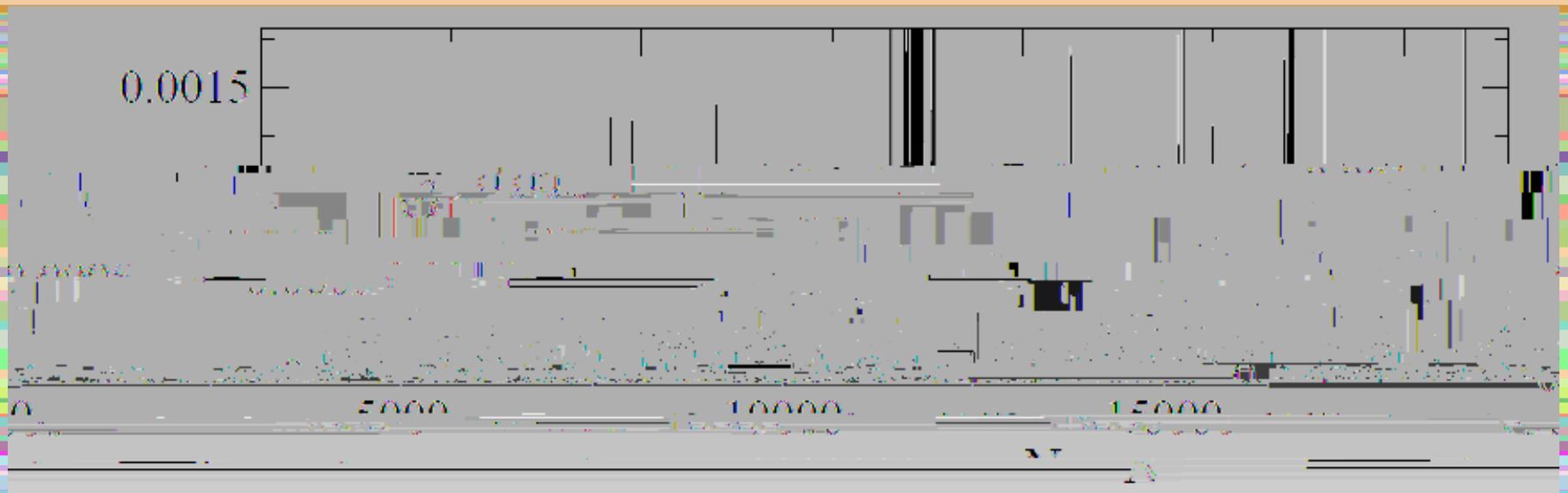




Example of a fractal eigenfunction

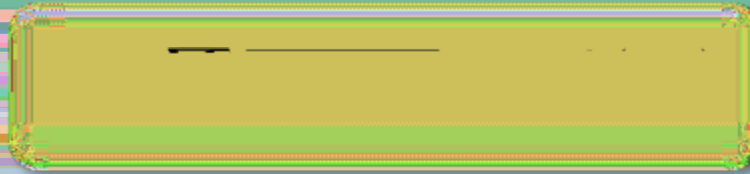


Outline

1. Power-law banded matrices and ultrametric ensemble: universality of the fractal dimensions
2. Fractal dimensions: beyond the universality
3. Dynamical scaling: validity of Chalker's ansatz

Fractal dimensions

Moments:



Metal: $d_q = d$

Insulator: $d_q = 0$

Anomalous scaling exponents: $\Delta_q \equiv (d_q - d)(q - 1)$

Critical point: $\Delta_q \neq 0$

Spatial correlations : $\overline{|j^2(t) - j^2(t')|} \propto (|t - t'|/E)^{\Delta_2}$

How one can calculate d_q ?

Green's functions:

$$I_{m+n} \iff \overline{(GR(\mathbf{r}, \mathbf{r}))^m (GA(\mathbf{r}, \mathbf{r}))^n}$$

Power-law banded random matrices

$$H_{ij} = \frac{1}{1 + (|i - j|/b)^{2\alpha}}$$



~ 1 critical states at all values of b

Almost diagonal matrices

Diagonal approximation: $\overline{I_q^{(0)}} = 1$

2x2 matrix approximation:

$$H(n, m) = \begin{pmatrix} H_{nn} & H_{nm} \\ H_{nm}^* & H_{mm} \end{pmatrix} \quad \overline{I_q^{(1)}} = \frac{1}{L} \sum_{n \neq m}^L (\overline{I_q(n, m)} - 1)$$

$$\overline{I_q} = 1 - \sqrt{\frac{\pi}{2}} \frac{\Gamma(q - 1/2)}{\Gamma(q - 1)} \frac{1}{L} \sum \sqrt{|H_{nm}|^2}$$

determines the nature of eigenstates
in the thermodynamic limit $L \rightarrow \infty$

$S \rightarrow \text{const}$

$S \sim L^\alpha, \quad \alpha > 0$

$\alpha = 1 - \nu$

localized states

extended states

critical states

Strong multifractality in the power-law ensemble

$$b \ll 1$$

General expression:

$$\overline{I}_q = 1 - \sqrt{\frac{\pi}{2}} \frac{\Gamma(q - 1/2)}{\Gamma(q - 1)} \frac{1}{L} \sum_{n=1}^L \sqrt{|H_{nm}|^2}$$

Power-law banded matrices:

$$|H_{nm}|^2 = \frac{1}{(\ln |n - m|/b)^2}$$

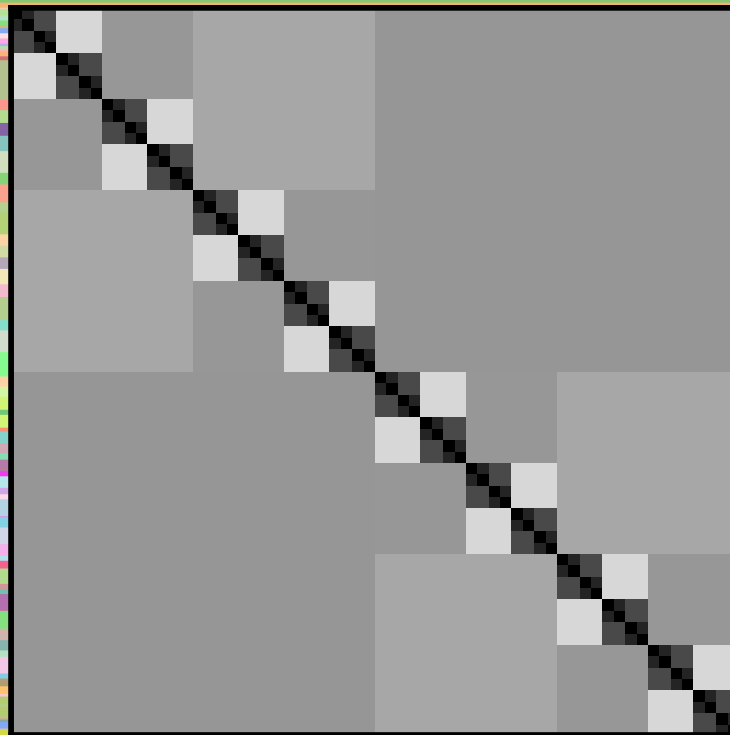
$$\overline{I}_q = 1 - \sqrt{\frac{\pi}{2}} \frac{\Gamma(q - 1/2)}{\Gamma(q - 1)} d_q \ln L$$

$$\overline{I}_q \propto L^{-d_q(q-1)} = 1 - d_q(q-1) \ln L + \frac{d_q^2(q-1)^2}{2} \ln^2 L + \dots$$

Fractal dimensions:

$$d_q = \sqrt{2\pi} \frac{\Gamma(q - 1/2)}{\Gamma(q)} b$$

Structure of the ultrametric matrix



Metal-insulator transition in the ultrametric ensemble

$$J/W \ll 1$$

General expression:

$$\overline{I}_q = 1 - \sqrt{\frac{\pi}{2}} \frac{\Gamma(q - 1/2)}{\Gamma(q - 1)} \frac{1}{L} \sum_{n \neq m}^L \frac{\sqrt{|H_{nm}|^2}}{W}$$

Ultrametric ensemble:

$$|H_{nm}|^2 = \frac{J^2}{L^{d(n,m)-2}}$$

$$\overline{|H_{nm}|^2} = \frac{1}{L} \sum_{n \neq m} \frac{J^2}{L^{(2/p) \log_2 L}} = \frac{J^2}{L^{(2/p) \log_2 L - 1}} \equiv \frac{1}{S}$$

$p > 2$	$S \rightarrow \text{const}$
$p < 2$	$S \sim L^\alpha, \quad \alpha > 0$
$p = 2$	$S \sim L^{-1}$

localized states
extended states
critical states

$$\overline{I}_q = 1 - \sqrt{\frac{\pi}{2}} \frac{\Gamma(q - 1/2)}{\Gamma(q)} \frac{1}{L} \sum_{n \neq m} \frac{\sqrt{|H_{nm}|^2}}{W}$$

Fractal dimensions in the ultrametric ensemble



Anomalous exponents:

$$\Delta_q \equiv (d_q - d)(q - 1)$$

Symmetry relation:

$$\Delta_q = \Delta_{1-q}$$

Y. V. Fyodorov, A. Ossipov and A. Rodriguez, J. Stat. Mech., L12001 (2009)

A. D. Mirlin et. al., Phys. Rev. Lett. **97**, 046803 (2007)

Universality of fractal dimensions

$$b = J/W \ll 1$$

Power-law banded matrices:

$$\overline{I_q} = \frac{\sqrt{\pi} \Gamma(q - 1/2)}{\Gamma(q) \sqrt{2\pi}}$$

Ultrametric random matrices:

$$\overline{I_q} = \frac{\sqrt{\pi} \Gamma(q - 1/2)}{\Gamma(q) \sqrt{2 \ln 2}}$$

$$\overline{I_q} = 1 - \sqrt{\frac{\pi}{2}} \frac{\Gamma(q - 1/2)}{\Gamma(q - 1)} \frac{1}{L} \sum_{n \neq m}^L \frac{\sqrt{|H_{nm}|^2}}{W}$$

universality

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Fractal dimensions: beyond universality

$$d_{\text{eff}}(b) = \frac{1}{\ln b} [f_1(a) + b f_2(a) + \dots]$$

$f_1(q)$ can be chosen the same for all models

Can we calculate $f_2(q)$?

Weak multifractality

$$b = J/W \gg 1$$

$$I_q^{(1)}(L) = -\frac{1}{b} F_1(q) \ln L + \dots$$

How one can calculate $F_i(q)$?

Mapping onto the non-linear β -model

Perturbative expansion in the regime $b \gg 1$

$$I_q^{(1)}(L) = -\frac{q!}{L} \left(1 + I^{(1)}(L) + I^{(2)}(L) + \dots \right) \quad I_q^{(1)} \propto \frac{1}{b}, \quad I_q^{(2)} \propto \frac{1}{b^2}$$

$$I_q^{(1)}(L) = -F_1(q) \frac{1}{b} \ln L + \dots \quad I_q^{(2)}(L) = \frac{F_1(q)^2}{b^2} \ln^2 L + \dots$$

Non

Outline

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Spectral correlations

$$C(\omega) = \overline{|\psi_n(\mathbf{r})|^2 |\psi_m(\mathbf{r})|^2} \propto \omega^{\nu-1} \quad \omega = E_n - E_m$$



$$I_2 = \sum_r |\psi_n(\mathbf{r})|^4 \propto L^{-d_2}$$

Strong multifractality: $d_2 \rightarrow 0 \Rightarrow \nu - 1 \rightarrow -1$



Strong overlap of two infinitely sparse fractal wave functions!

Return probability

$$P(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} C(\omega)$$

$$C(\omega) \propto \omega^{\nu-1} \iff P(t) \propto \frac{1}{(bt)^\nu}$$

$$\lim_{L \rightarrow \infty} \frac{P(t)}{L^{-d_2}} \propto 1$$

Strong multifractality $d_2, \nu \ll 1$:

$$\ln P(t) \approx -\nu \ln(bt) + \frac{\nu^2}{2} \ln^2(bt) + \dots \quad bt \ll L$$

$$d_2 = \frac{\ln 4 + \pi I}{\ln 2} + O(\nu^3) \approx \frac{\pi \bar{b}}{\sqrt{2}} + \frac{(\pi b)^2}{4\sqrt{2}} \left| 10 \frac{1}{2\sqrt{2}} - \frac{56}{1} \right|$$

