Universality in complex Wishart ensembles

M. Mo: arXiv:0809.   0, arXiv:08   . 8

Eigenvalue statistics of complex Wishart ensembles

Multiple Laguerre polynomials and Riemann-Hilbert problem

Results

$_N$ has distinct eigenvalues, N_0 of them and N of them a.

M N and as $N \rightarrow$, both N_0, N $\rightarrow$ and $\frac{N}{M} \rightarrow c$, $\frac{N}{N} \rightarrow$.

Universality: Eigenvalue correlations given by the Sine-kernel in the bulk of the spectrum and Airy kernel in the lim

Global eigenvalue statistics

Studied by Bai, Choi and Silverstein with Stieltjes transform.

$$m_G(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dG(x), \quad z \in \mathbb{C}^+ \equiv \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$$

The Stieltjes transform of the limiting eigenvalue distribution satisfies

$$m(z) = \int_{\mathbb{R}} \frac{1}{t(1 - c - czm) - z} dH(t)$$

$H(t)$: limiting eigenvalue distribution of Σ_N.

Local eigenvalue statistics

Most results assumed the covariance matrix Σ_N is a finite rank perturbation of the identity matrix.

This is one of the few results for non-spiked models.

Baik, Ben-Arous and Péché (0): eigenvalue correlation functions in terms of a determinantal formula.

Multiple Laguerre polynomials

Generalization of Laguerre polynomials that are orthogonal to multiple weights: $x^{M-N}e^{-Ma_j x}$. E.g.

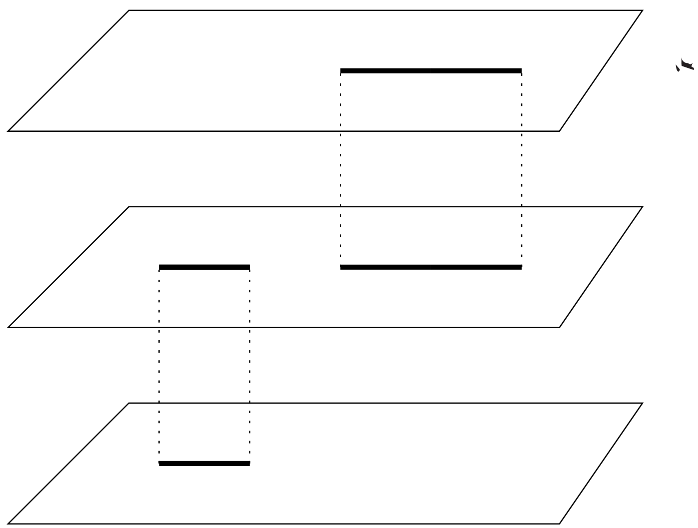
$$\int_0 L_{n_1,n}(x)x^{i+M-N}e^{-Mx}dx = 0 \quad i = 0, \ldots, n_1 - 1$$

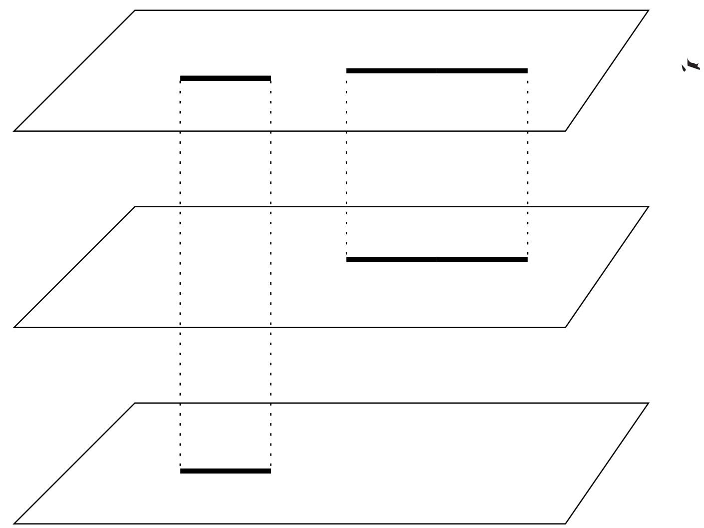
$$\int_0 L_{n_1,n}(x)x^{i+M-N}e^{-Max}dx = 0 \quad i = 0, \ldots, n - 1$$

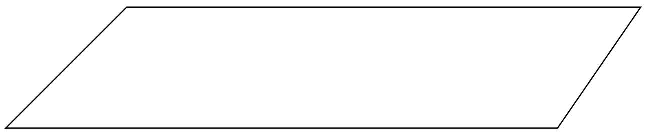
(Bleher and Kuijlaars, Desrosiers and Forrester) Correlation kernel of complex Wishart ensembles can be expressed in terms of multiple Laguerre polynomials.

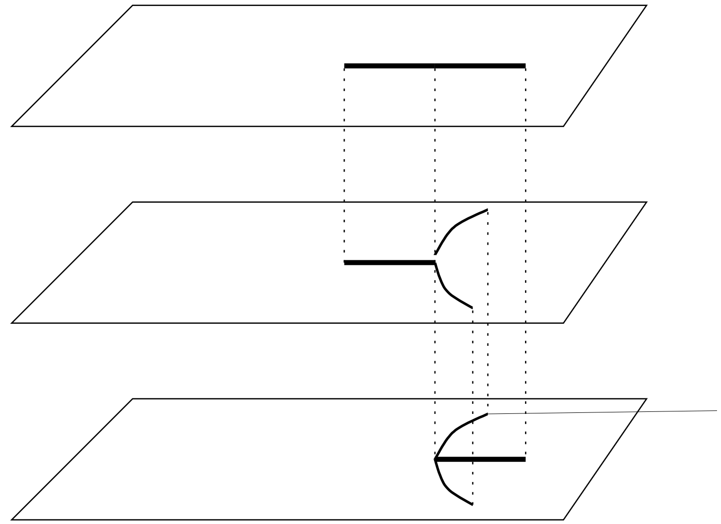
Multiple Laguerre polynomials are solutions of Riemann-

Riemann surface depends on parameters and di cult to determine sheet structure in this case.









Stieltjes transform

De ne $\underline{F}$ by

$$\underline{F} = (- c) I_{0}) + cF$$

then the Stieltjes transform $m_{\underline{F}}$ satis es

$$m_{\underline{F}}(z) = - \left(z - c \quad_{\mathbb{R}} \frac{t dH(t)}{ + t m_{\underline{F}}} \right)$$

In our case, $dH(t)$ is

$$dH(x) = (-) + \quad a$$

Gives us an algebraic equation

$$z = \quad - + c \frac{-}{+} + c \frac{a}{+ a}$$

Observations:

. The equation has solutions behaves like

$$(z) \quad -\frac{}{z} + O(z^{-}) \qquad z \to$$

$$(z) \quad + \frac{c(\; - \;)}{z} + O(z^{-}) \qquad z \to$$

$$(z) \quad -\frac{}{a} + \frac{c}{z} + O(z^{-}) \qquad z \to$$

. $m_{\underline{F}}$ is the solution . So has branch cut on the support of $\underline{F}$.

Lemma 1 If z . supp($\underline{F}$)**, then** $m \equiv m_{\underline{F}}(z)$ **satisfies the following.**

1. m $\mathbb{R}$ 0 **;**

2. $-\frac{1}{m}$. supp(H)**;**

3. $z'(m)$ 0**.**

Conversely, if m **satisfies 1-3, then** $z \equiv z(m)$. supp($\underline{F}$)**.**

$z' \equiv 0$ are the potential end points of the support.

Zeros of the polynomials

$$a \quad (\quad - c) \quad + \quad (a \, (\quad - c \quad) + a(\quad - c(\quad - \quad))$$
$$+ \quad (\quad - c(\quad - \quad) + a \, (\quad - c \quad) + \quad a) \quad + \quad (\quad + a) \quad + \quad = 0$$

Discriminant positive, real roots , Discriminant negative, real roots .

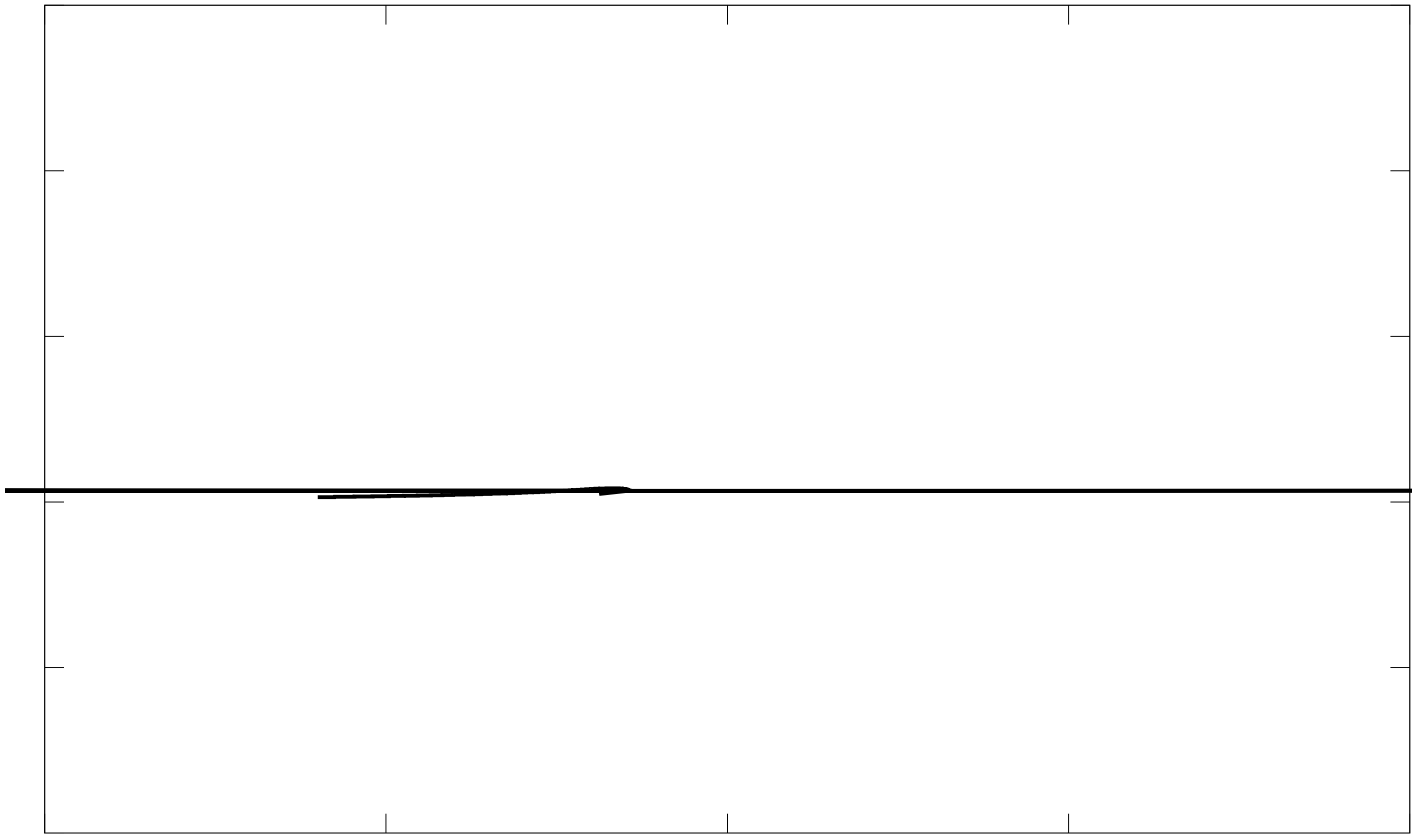
Complement of support: E.g. when there are real roots,

$$\text{supp}(\underline{F})^c = (- \quad 0) \quad (0 \quad) \quad (\quad) \quad (\quad)$$

$_k$ are given by

$$_k = \quad - \frac{}{_k} + c \frac{-}{+ \quad _k} + c \frac{a}{+ a \quad _k}$$

So if we have , then support consists of intervals, otherwise support has interval.



Summary

Studied complex Wishart ensembles whose covariance matrix has N_0 eigenvalues and N eigenvalues a and $\frac{N}{M} \to c$, $\frac{N}{N} \to$.

One of the few cases when results was obtained for non-spiked models.

Correlation kernel given by Sine-kernel in bulk and Airy kernel in edge. Tracey-Widom distribution for largest eigenvalue.

Uses Stieltjes transform to overcome di culties in the application of Riemann-Hilbert analysis.

When covariance matrix has nitely many distinct eigenvalues, Stieltjes transform gives a Riemann surface and the analysis can be generalized to the case when the Riemann surface has less than or equal to complex branch points.