Universality of Local Bulk Regime for Hermitian Matrix Models

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1 Introduction

1.1 Asymptotic ”Philosophy” of RMT

Let M_n be a n S σ 2G - 93 1 0 . 9 9 1 c T 2 F 4 1 . 4 1 1 6 6 9 1 . [(n T

(ii) $\overline{N}_n \to N$ weakly as $n \to \infty$;

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(iii) $N(\ ) = \int (\ ) d :$

Property (ii) fixes the global scale of the spectral axis, yielding

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i.e., $(nN(\ ))^{-1}$ is the typical eigenvalue spacing for large n in .

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Probability theory analogs: LLN, CLT, ”collective theorems”, *Yu. Linnik*

”emerging universality” QFT

1.2 Linear Eigenvalue Statistics.

Take $\varphi : \mathbb{R} \to \mathbb{R}$ and write the *linear eigenvalue statistic*

$$\mathcal{N}_n[\varphi] := \sum_{l=1}^{n} \varphi(\lambda_l^{(n)}) = \operatorname{Tr} \varphi(M_n) = \int_{\mathbb{R}} \varphi(\lambda) N_n(d\lambda).$$

φ is known as the *test function*. In particular

$$N_n(\Delta) := \sharp\{\lambda_l^{(n)} \in \Delta,\ l = 1, \ldots, n\} = \sum_{l=1}^{n} \chi_\Delta(\lambda_l^{(n)}) = \mathcal{N}_n[\chi_\Delta]$$

is the Eigenvalue Counting Measure of eigenvalues and $\overline{N}_n = n^{-1} N_n$.

Define

$$\text{bulk}\, N = \{ \ \in \operatorname{supp} N : \exists \ > 0; \lim_{n \to \infty} \sup_{| \ |} | \ (\ ) \ \ _n(\ )| = 0\}.$$

We have for $N_n[\varphi]$:

- φ is n-independent: global regime;
- $\varphi_n = \varphi((\ \ \ _0)L_n); \ L_n \to \infty; \ nL_n \to 0$: intermediate bulk regime;
- $\varphi_n = \varphi((\ \ \ _0)n \ \ _n(\ _0))$: local bulk regime

1.3 Typical Problems

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(ii) $\mathrm{Var}\{N_n\}$, global and intermediatee regime, fluctuations, "CLT"

(iii) $\mathbf{P}\{N_n(\Delta_n) = k\};\ k \in \mathbb{N};\ E_n(\Delta_n) = \mathbf{P}\{N_n(\Delta_n) = 0\}$,

gap probability, local regime, spacings, universality

in particular $\Delta_n = (\lambda_0; \lambda_0 + s/n\rho_n(\lambda_0));\ \lambda_0 \in \text{bulk } N$ for the local bulk regime

1.4 Hermitian Matrix Models

$n \times n$ hermitian random matrices with the law

$$\mathrm{P}_n(dM) = Z_n^{-1} \exp\{-n \operatorname{Tr} V(M)\} dM,$$

$$dM = \prod_{j=1}^{n} dM_{jj} \prod_{1 \le j < k \le n} d\Re M_{jk} \, d\Im M_{jk},$$

$V : \mathbb{R} \to \mathbb{R}_+$ is a continuous function (potential), and

$$\exists \varepsilon > 0, \; L < \infty \quad V(\lambda) \ge (2+\varepsilon)\log(1+|\lambda|) > 0, \; |\lambda| \ge L$$

$V = \lambda^2/2$ corresponds to the Gaussian Unitary Ensemble (GUE).

1.5 Results (a collection)

(i) For any probability measure m on $\mathbb{R}$; $m(\mathbb{R}) = 1$ define (*Gauss*)

$$E[m] = \int V(\ )m(d\ ) \qquad \int\int \log| \qquad |m(d\ )m(d\ );$$

and let N

In particular

$$\mathrm{v.p.}\int_{\mathrm{supp}\ N} \frac{(\quad)d}{} = V'(\quad)/2; \quad \in \mathrm{supp}\ N.$$

i.e., an analog of the LLN : *Wigner, 52; Brezin et al, 79; A. Boutet de Monvel, P., Shcherbina, 95; Deift et al 98; Johansson, 98; P., Shcherbina, 07*.

(ii) $\mathrm{Var}\{N_n[\varphi]\}$ does not grow with n if $\varphi \in$

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V^{0} *is Lip 1;*

there exists a closed interval $[a; b] \qquad = \mathrm{supp}\, N$ *such that*

$$\sup_{2[a;b]} jV^{000}(\ )j \quad C_1 < 1; \quad 0 < \inf_{2[a;b]} (\ ):$$

Then we have for any $d > 0$:

(i)

$$\sup_{2[a+d;b \quad d]} j \ {}_n(\ ) \qquad (\ )j \quad Cn^{\ 2=9};$$

i.e., $[a + d; b \qquad d]$ *belongs to bulk* N ;

(ii) if $p_l^{(n)};\; l = 1; 2; :::$ *are the marginals of the joint probability density of eigenvalues, then for any* $_0 \; 2 \; [a + d; b \quad d]$

$$\lim_{n \,!\, 1} [\;_n(\;_0)]\;\; ^{l}p_l^{(n)} \quad _0 + \frac{x_1}{n \;\; _n(\;_0)}; :::; \quad _0 + \frac{x_l}{n \;\; _n(\;_0}$$

(iii) *if* $E_n(\quad _n) = \mathrm{P} f \; {}^{(n)}_{l} \; 2 \quad _n; \; l = 1; :::; ng$ *is the gap probability of the ensemble and* $_n = (\; _0; \; _0 + s{=}n \; (\; _0)) \quad _0 \; 2 \; [a + d; b \quad d]$, *then*

$$\lim_{n \,!\, 1} \mathrm{P}(\quad _n) = \det(1 \quad S(s));$$

where

$$(S(s)f)(x) = \int_0^s \frac{\sin \quad (x \quad y)}{(x \quad y)} f(y)dy; \;\; x \; 2 \; [0; s]:$$

Dyson, 61, 73; P., Shcherbina, 97, 07; Deift et al 99

More if $(\; _0) = 0; \; 1$ (singular points (e.g. edge) universality).

2 Proof (outline)

2.1 Orthogonal Polynomials Techniques

Weyl integration formula for the joint eigenvalue density

$$p_n(\;_1;\ldots;\;_n) \quad = \quad Q_n^{\;1} e^{\;\;n\;\sum_{k=1}^{n} V(}$$

2 Proof (outline)

2.1 Orthogonal Polynomials Techniques

Weyl integration formula for the joint eigenvalue density

$$p_n$$

$\;_l^{(n)} = e^{\;\;nV=2}P_l^{(n)}$ and $K_n(\;\;;\;\;) = \sum_{l=0}^{n\;\;1}\;_l^{(n)}(\;\;)\;_l^{(n)}(\;\;)$,

$$\int K_n(\;\;;\;\;)K_n(\;\;;\;\;)d\;\; = K_n(\;\;;\;\;):$$

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$$ \int K_n(\;\; ; \;\;) K_n(\;\; ; \;\;) d \;\; = K_n(\;\; ; \;\;): $$

Then the marginals $p_l^{(n)}$ of p_n are given by the determinant formulas

$$ p_l^{(n)}(\;\; {}_1; :::; \;\; {}_l) : = \int p_n(\;\; {}_1; :::; \;\; {}_n) d \;\; {}_{l+1} ::: d \;\; {}_n $$

$$ = (n:::(n \quad l + 1)) \;\; {}^1 \det f K_n(\;\; {}_j ; \;\; {}_k) g_{j;k=1}^l : $$

Determinant formulas imply:

(a) $\mathrm{E} f n \;\; {}^1 N_n['] g = \int \; '(\;\;) \;\; {}_n(\;\;) d \;\; ; \quad {}_n(\;\;) = K_n(\;\; ; \;\;);$

(b) $\mathrm{Var}\{N_n[\varphi]\} = \frac{1}{2}\int\int(\varphi(\lambda_1) - \varphi(\lambda_2))^2 K_n^2(\lambda_1;\lambda_2)\,d\lambda_1 d\lambda_2;$

(b) $\mathrm{Var}\, f N_n[\,'\,]g = \frac{1}{2}$

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(c) $\mathrm{P}\{N_n(\ ) = 0\} = \det(1 \quad K_n(\ ))$, where

$$(K_n(\ )f)(\ ) = \int K_n(\ ;\ ) f(\ ) d\ ; \quad 2 \quad :$$

Thus, problems ((i)) - (ii) - (iii) reduce to the asymptotic analysis of the reproducing kernel K_n,

or, in view of the *Christoffel-Darboux* formula

$$K_n(\ ;\ ) = r^{(n)}_{n\ 1} \quad {}^{(n)}_{n}(\ )\ {}^{(n)}_{n\ 1}(\ ) \quad {}^{(n)}_{n\ 1}(\ )\ {}^{(n)}_{n}(\ )\ (\qquad)\ ^{1};$$

to the asymptotics of ${}^{(n)}_{n};\ {}^{(n)}_{n\ 1}$, and $r^{(n)}_{n\ 1}$, where for $l \quad 0$.

$${}^{(n)}_{l}(\ ) = r^{(n)}_{l}\ {}^{(n)}_{l+1}(\ ) + s^{(n)}_{l}\ {}^{(n)}_{l}(\ ) + r^{(n)}_{l\ 1}\ {}^{(n)}_{l\ 1}(\ ):$$

These asymptotics were found by *Deift et al, 99* and lead to the solution of (iii), i.e., the universality of the local bulk regime of the hermitian matrix models, and (ii), i.e., the limiting laws of fluctuations of linear eigenvalue statistics, the CLT in particular, *P., 06*, for real analytic potentials.

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On the other hand, in *P., Shcherbina 97, 07* the universality of the local bulk regime of hermitian matrix models is proved for globally C^2 and locally C^3 potentials (see above theorem), basing on the orthogonal polynomial techniques, in particular on the above integral representation for K_n, but NOT using asymptotics of corresponding orthogonal polynomials.

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In *P., Shcherbina, 97* $\sin(\ x)=(\ x)$ is obtained via its Taylor expansion.

In *P., Shcherbina, 07* $\sin(\ x)=(\ x)$ is obtained as solution of a non-linear integro-differential equation.

2.2 Integro-differential Equation for Rescaled Reproducing Kernel

We start from the integral representation à la determinant formulas

$$n^{-1}K_n(\xi;\eta) = Q_{n;2}^{-1} e^{-n(V(\xi)+V(\eta))/2} \int \prod_{j=2}^{n} d\lambda_j e^{-nV(\lambda_j)}(\xi-\lambda_j)(\eta-\lambda_j) \prod_{2\le j<k\le n} (\lambda_j-\lambda_k)^2$$

2.2 Integro-differential Equation for Rescaled Reproducing Kernel

We start from the integral representation `

Differentiate the representation with respect to x to obtain the identity

$$
\begin{aligned}
\frac{\partial}{\partial x} K_n(x,y) = &\quad \frac{1}{2} V'(\ _0 + x/n) K_n(x,y) \\
&+ \int \frac{K_n(x',x')K_n(x,y) \quad K_n(x,x')K_n(x',y)}{x \quad x'} dx'.
\end{aligned}
$$

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$$+ \int \quad K_n(x';x')K_n(x;y) \quad K_n(x;x'$$

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We prove next that under the conditions of theorem we have uniformly in

$|x|, |y| < L, \ \theta_0 \in [a + d, b - d]$:

$$\frac{\partial}{\partial x} K_n(x, y) + \frac{\partial}{\partial y} K_n(x, y) \le C \, n^{-1/8} + |x - y| n^{-2} ;$$

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$$\left|\frac{\partial}{\partial x} K_n(x,y) + \frac{\partial}{\partial y} K_n(x,y)\right| \le C\left(n^{-1/8} + |x-y| n^{-2}\right),$$

$$|K_n(x,y) - K_n(0, y-x)| \le C|x|\left(n^{-1/8} + |x-y| n^{-2}\right),$$

$$\left|\frac{\partial}{\partial x} K_n(x,y)\right| \le C; \quad \int_{|x| \le L} dx \left|\frac{\partial}{\partial x} K_n(x,y)\right|^2 \le C.$$

We prove next that under the conditions of theorem we have uniformly in $|x|, |y| < L,\ \lambda_0 \in [a+d, b-d]$:

$$\left|\frac{\partial}{\partial x}K_n(x;y)+\frac{\partial}{\partial y}K_n(x;y)\right| \le C\left(n^{-1/8}+|x-y|n^{-2}\right);$$

$$|K_n(x;y)-K_n(0;y-x)| \le C|x|\left(n^{-1/8}+|x-y|n^{-2}\right);$$

$$\left|\frac{\partial}{\partial x}K_n(x;y)\right| \le C; \quad \int_{|x|\le L} dx\left(\frac{\partial}{\partial x}K_n(x;y)\right)^2 \le C.$$

Now, if

$$
\begin{aligned}
K_n(x) &= K_n(x;0)\mathbf{1}_{|x|\le L} + K_n(L;0)(1+L-x)\mathbf{1}_{L<x\le L+1} \\
&+ K_n(-L;0)(1+L+x)\mathbf{1}_{-L-1\le x<-L};
\end{aligned}
$$

then for $|y| \le L$

$$
\frac{\partial}{\partial y}K_n(y) = \int_{|x'|\ge 2L/3} \frac{K_n(x')K_n(y-x')}{x'}\,dx' + O(L^{-1});
$$

and $\displaystyle\int |K_n(x)|^2 dx \le 1$; $\displaystyle\int \left(\frac{d}{dx}K_n(x)\right)^2 dx \le 1.$

2.3 Asymptotic Solution of Equation

Consider the Fourier transform

$$\hat{K}_n(p) = \int K_n(x)e^{ipx}dx; \quad K_n(x) = (2\ )^{-1}\int \hat{K}_n(p)e^{\ ipy}dp.$$

Then we have from $n\ ^{1}K_n(\ ;\ \ ) =$

Since K_n is "asymptotically even"

$$\int |\hat{K}_n(p) - \hat{K}_n(-p)|^2 dp$$

$$= 2 \int |K_n(x) - K_n(-x)|^2 dx \le C n^{-1/8} \log^3 n.$$

we obtain the Fourier form of the above integro-differential equation:

$$\int \hat{K}_n(p) \int_0^p \hat{K}_n(p')dp' \cdot p \, e^{-ipy} dp = O(L^{-1}); \; |y| \ge L/3.$$

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we obtain the Fourier form of the above integro-differential equation:

$$\int \hat{K}_n(p) \int_0^p \hat{K}_n(p') dp' - p \, e^{-ipy} dp = O(L^{-1}); \; |y| \le L/3.$$

Besides, since K_n is positive definite $\hat{K}_n$ is ”asymptotically non-negative”:

$$\int \hat{K}_n(p) |\hat{f}(p)|^2 dp \ge -C \|f\|^2_{L^2(\mathbb{R})} (n^{-1/8} \log^4 n + O(L^{-1})).$$

Now consider the functions

$$F_n(p) = \int_0^p \tilde{K}_n(p')\,dp'.$$

Since $p\tilde{K}_n \in L^2(\mathbb{R})$, the sequence $\{F_n\}$ consists of functions that are of uniformly bounded variation, uniformly bounded and equicontinuous on $\mathbb{R}$.

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Now consider the functions

$$F_n(p) = \int_0^p \hat{K}_n(p')\,dp'.$$

Since $p\hat{K}_n \in L^2(\mathbb{R})$, the sequence $\{F_n\}$ consists of functions that are of uniformly bounded variation, uniformly bounded and equicontinuous on $\mathbb{R}$. Thus $\{F_n\}$ is a compact family with respect to the uniform convergence. Hence, the limit F of any subsequence $\{F_{n_k}\}$ possesses the properties:

(a) F is bounded and continuous;

(b) $F(p) = \quad F(\quad p)$;

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$$F_n(p) = \int_0^p \widehat{R}_n(p')\,dp'.$$

Since $\sqrt{\widehat{R}_n} \in L^2(\mathbb{R})$, the sequence $\{F_n\}$ consists of functions that are of uniformly bounded variation, uniformly bounded and equicontinuous on $\mathbb{R}$. Thus $\{F_n\}$ is a compact family with respect to the uniform convergence. Hence, the limit F of any subsequence $\{F_{n_k}\}$ possesses the properties:

(a) F is bounded and continuous;

(b) $F(p) = -F(-p)$;

(c) $F(p) \le F(p')$, if $p \le p'$;

Now consider the functions

$$F_n(p) = \int_0^p \hat{\varphi}_n(p')\,dp'.$$

Since $p\hat{\varphi}_n \in L^2(\mathbb{R})$, the sequence $\{F_n\}$ consists of functions that are of uniformly bounded variation, uniformly bounded and equicontinuous on $\mathbb{R}$. Thus $\{F_n\}$ is a compact family with respect to the uniform convergence. Hence, the limit F of any subsequence $\{F_{n_k}\}$ possesses the properties:

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(d) $F(+\infty) - F(-\infty) = 2\,(\quad_0)$;

Now consider the functions

$$F_n(p) = \int_0^p \tilde{k}_n(p')\,dp'.$$

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(c) $F(p) \quad F(p')$, if $p \quad p'$;

(d) $F(+\infty) \quad F(\quad \infty) = 2 \quad (\quad {}_0)$;

(e) F satisfies the following equation, valid for any smooth function g of compact support:

$$\int (F(p) \quad p)g(p)dF(p) = 0.$$

The last property implies that $F(p) = p$ or $F(p) = \text{const}$, hence it follows from (a) – (c) that

$$F(p) = p\,1_{|p| \quad p_0} + \quad \mathrm{sign}(p)\,1_{|p| \quad p_0},$$

where $p_0 = \quad (\ _0)$ by (d).

We conclude that the equation is uniquely soluble, thus the sequence $\{F_n\}$ converges uniformly on any compact to the above F. This imply the weak convergence of the sequence fK_ng to the function

$$K\ \ (x) = \frac{\sin(\quad(\ _0)x)}{(\ _0)x}:$$

convergence of the sequence $\{K_n\}$ to the function

$$K\ (x) = \frac{\sin(\ \ (\ _0)x)}{(\ _0)x}.$$

But weak convergence implies

$$\lim_{n \to \infty} K_n(x, y) = K\ (x\ \ y).$$

uniformly in (x, y), varying on a compact set of $\mathbb{R}^2$, because

$$\frac{d}{dx} K_n \in L^2(\mathbb{R}).$$