Entanglement entropy in quantum spin chains with nite range interaction

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B

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Fundamentals and Applications
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Work in collaboration with Alexander Its and Man Yue Mo

Outline

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1. The problem
2.

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Questions we can ask:

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Mezzadri
determinants
and the RH
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quantum spin

The problem

Consider a one-dimensional quantum spin chain:

Look at the ground state $|\Psi_g\rangle\langle\Psi_g|$ ($T = 0$: phase transition in the thermodynamic limit.)

Questions we can ask:

- What is the entropy of the entanglement between A and B as $L \to \infty$?
- What is the correlation between two spins at different sites?
- Many others.

If the model is:

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Its, Mezzadri and Mo (2007):

Spin chains and block-Toeplitz determinants

The entropy of the entanglement can be written as

$$S(\rho_A) = \lim_{\epsilon \to 0^+} \lim_{L \to \infty} \frac{1}{4\pi i} \oint_{\Gamma(\epsilon)} e(1+\epsilon, \lambda) \left[\frac{d}{d\lambda} \log D_L(\lambda)(\lambda^2 - 1)^{-L} \, d\lambda \right];$$

Spin chains and block-Toeplitz determinants

Spin chains and block-Toeplitz determinants

$D_L(\ )$ is the determinant of the block-Toeplitz matrix

$$T_L[\ ] = \begin{pmatrix} B_0 & B_{-1} & \cdots & B_{2-L} & B_{1-L} \\ B_1 & B_0 & \cdots & B_{3-L} & B_{2-L} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{L-2} & B_{L-3} & \cdots & B_0 & B_{-1} \\ B_{L-1} & B_{L-2} & \cdots & B_1 & B_0 \end{pmatrix}$$

Spin chains and block-Toeplitz determinants

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$$g^2(z) = \prod_{j=1}^{2n} (z - z_j)$$

Spin chains and block-Toeplitz determinants

$$g^2(z) = \quad {}^2$$

Spin chains and block-Toeplitz determinants

Remarks:

The branch cuts of $g(z)$ are the segments

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$$L : w^2 = \prod_{i=1}^{4n} (z \qquad _i):$$

The genus of L is $g = 2n \quad 1$.

Statement of results

Define $\theta : \mathbb{C}^g \to \mathbb{C}$ associated to L by

$$\theta(\vec{s}) := \sum_{\vec{n} \in \mathbb{Z}^g} e^{i\pi \vec{n} \cdot \Omega \vec{n} + 2\pi i \vec{s} \cdot \vec{n}}.$$

Ω is a $g \times g$ symmetric matrix (period matrix) that depends on the Hamiltonian through the branch cuts of L.

Statement of results

What happens at a phase transition?

Block-Toeplitz determinants and the RH problem

How do we compute such formulae?

The symbol $(z) = \; i \; g^{-1}(z)$ ... $g(z)$ ... admits ... Wiener-Hopf

$(z) \; z =$

Block-Toeplitz determinants and the RH problem

How do we compute such formulae?

The symbol $(z) = \begin{pmatrix} i & g(z) \\ g \; ^{1}(z) & i \end{pmatrix}$ admits the Wiener-Hopf factorization:

$$(z) = U_{+}(z)U \;\; (z) = V \;\; (z)V_{+}(z);$$

where $U \;\; (z)$ and $V \;\; (z)$ are analytic inside/outside the unit circle and

$$U \;\; (1) = V \;\; (1) = I:$$

Block-Toeplitz determinants and the RH problem

Theorem (Widom, 1974)

$$\frac{d}{d} \log D_L(\quad) = \frac{2}{1 \qquad ^2} L + \frac{1}{2} \int_{S^1} \mathrm{tr} \Big[\qquad U_+' \Big]$$

Block-Toeplitz determinants and the RH problem

It turns out that

$$V\ (z) = {}_3U\ (z)^{\ 1}\ {}_3$$

$$V_+(z) = {}_3U_+(z)^{\ 1}\ {}_3(\ ^2\ \ 1); \qquad 6\ 1$$

Block-Toeplitz determinants and the RH problem

Block-Toeplitz determinants and the RH problem

Let us define

$$S(z) = U_{-}(z)Q(z)^{-1}, \qquad |z| \geq 1;$$
$$S(z) = U_{+}(z)^{-1}Q(z), \qquad |z| \leq 1.$$

In order to compute the entropy of entanglement we need:

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1. to solve the previous RH problem for $S(z)$ in terms of $)\quad \vartheta 564$

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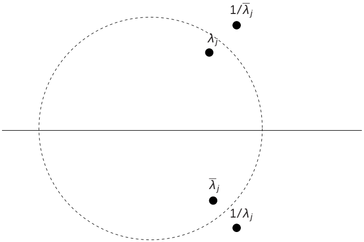
Pairs of roots of $g(z)$ approach the unit circle.



The period matrix in

$$(\vec{s}) := \sum_{\vec{n} \in \mathbb{Z}^g} e^{i \, \vec{n} \quad \vec{n} \quad 2i \, \vec{s} \quad \vec{n}}.$$

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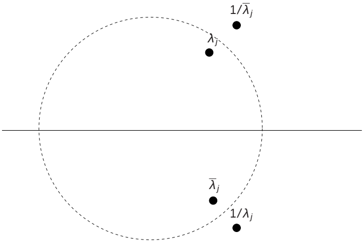

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$(\vec{s})$ becomes singular.

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Such determinants are computed by solving a RH problem.

At phase transition we observe logarithmic divergences that generalize previous results.

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- We computed the entropy of the entanglement of the ground state of integrable quantum spin chains with nite range and translation invariant interaction.
- At the core of the computation is the evaluation of block-Toeplitz determinants.
- Such determinants are computed by solving a RH problem.
- At phase transition we observe logarithmic divergences that generalize previous results.

AR Its, F Mezzadri and MY Mo. Entanglement entropy in quantum spin chains with nite range interaction. arXiv:0708.0161v1.