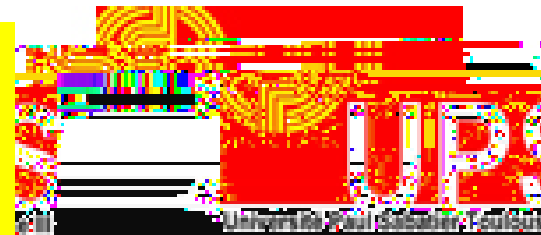


# Extreme Value Statistics of Maximal Eigenvalues of Random Matrices

David Dean

LPT Université Paul Sabatier  
Toulouse and Institut Universitaire  
de France



Work in collaboration with Satya Majumdar, Paris XI

PRL 97 160201 (2006)

# Plan of Talk

- Motivation - statistics and string theory
- Coulomb gas formulation for Gaussian ensembles
- Solution of problem by hard wall constraint- half Hilbert transform
- Some numerical tests

# Conditioned matrix ensembles

$M$  -random matrix e.g. from GOE, GUE or GSE ensemble

Say  $M$  describes Hamiltonian of vibrations about some stationary point in the energy landscape or the stability matrix of a fixed point in a dynamical system. Symmetry and complexity determine Gaussian statistics but in an experiment we can only observe systems where all eigenvalues are  $> z$

- (i) What is the probability of finding such a matrix ?
- (ii) What are its spectral properties ?

# Anthropic principles

The current form of string theory is believed to have  $10^{500}$  possible vacua -the Landscape-each corresponding to a possible Universe and physical constants.

It seems unlikely that life can evolve in a generic universe so we are lucky to be here - anthropic principle this is the only type of Universe we could possibly see, there could be lots of others (statistically dominating) which are quite different to ours.

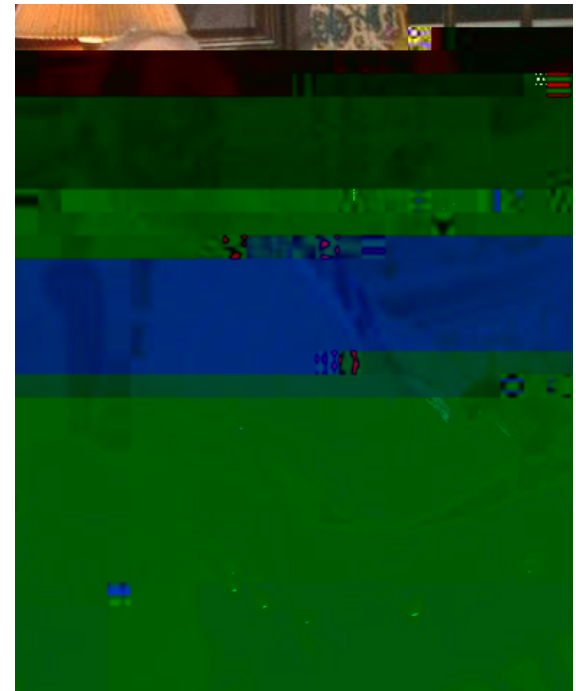
What properties of our universe depend of the details of string theory and which depend simply on the statistics of the complex/random landscape ?

# Anthropic Reasoning



Fred Hoyle- a  $^{12}\text{C}$  based life form reasoned there should be a mechanism that permits sufficient  $^{12}\text{C}$  for life to exist, he then found the  $^4\text{He}+^8\text{Be}$  resonance of  $^{12}\text{C}$  that is responsible

Stephen Weinberg: the cosmological constant is small but if one takes the maximal value avoiding a « Big Rip » this gives a surprisingly good estimate



# Estimating the probability of a stable universe


Aazami and R. Easther, J. Cosmol.Astropart. Phys. JCAP03 013 (2006)- classical string landscape is large dimensional (N)

If M describes the Hessian of a critical point what is the probability that it is a minimum, i.e. that all the eigenvalues are positive if M is chosen from GOE :

$$P(\lambda_{\min} > 0) \approx \exp(-N^2 \theta(0))$$

Aazami and Easther found an approximation (GOE) that  $\theta(0)=1/4$  and numerically  $\theta(0) = 0.3291$  diagonalizing up to  $N = 7$  matrices.

But Gaussian random potential model is quite different-  
AJ Bray and DD PRL 2007



QuickTime™ and a  
TIFF (Uncompressed) decompressor  
are needed to see this picture.

# Coulomb Gas Formulation

$$P(\{\lambda_i\}) = B_N \exp \left[ -\frac{\beta}{2} \sum_{i=1}^N \lambda_i^2 \right] \prod_{i < j} |\lambda_i - \lambda_j|$$



# Hard Wall Constraint

$P_N$

# Density functional method

$$f(\mu) = \frac{1}{N} \sum_{i=1}^N \delta(\mu - \mu_i)$$

Normalised density field

$$H[f] = -N^2 \Sigma[f]$$

Superextensive energy scaling

$$\Sigma[f] = -\frac{1}{2} \int f(\mu) \mu^2 d\mu + \frac{1}{2} \int f(\mu) f(\mu') \ln(|\mu - \mu'|) d\mu d\mu'$$

Constraints on  $f$ :

$$f(\mu) = 0 \text{ for } \mu < z \quad ; \quad \int f(\mu) d\mu = 1$$

$$Z_N(z) = \int d[f] J[f] \exp(N^2 \beta \Sigma[f])$$

Jacobian to pass from coordinates  
To density field

# Functional Integral

$$\begin{aligned} \mathcal{J}[f] &= C_N \int \prod_i d\mu_i \delta(Nf(\mu) - \sum_i \mu_i) \\ &= C_N \int d[g] \exp(N \int f(\mu)g(\mu) + N \ln(\int \exp(-g(\mu))d\mu)) \end{aligned}$$

Jacobian term is  $\exp(O(N))$  and is thus negligible with respect to the energy term  $\exp(O(N^2))$ .

Formally  $\mathcal{J}[f] = C_N'' \exp\left(-N \int d\mu f(\mu) \ln(f(\mu))\right)$

If large N saddle-point can be justified above - mean field like entropy term (slightly more to it than that).

# Leading order behaviour

$$Z_N(z) = D_N \exp(N^2 \beta S(z) + O(N))$$

$$S[z] = \max_f \{ \Sigma[f] \}$$

Saddle point equation,  
gives thermodynamically

$$\mu = P \int_z^{\infty} d\mu' \frac{f(\mu)}{\mu - \mu'}$$

$$\mu = z + x \Rightarrow z + x = (H_+ f)(x)$$

$$H_+ F = P \int_0^{\infty} dx' \frac{F(x')}{x - x'}$$



# Recovering Wigner's Semi-Circle Law

However  $f$  must be real and positive, but  $f(0) < 0$  if  $L(z) < -2z$

$$\Rightarrow z < -\sqrt{2}$$

$$f(\mu) = \frac{1}{\pi} \sqrt{2 - \mu^2}$$

Wigner semi-circle law when  $z = -\sqrt{2}$  - but this solution is Ok for all  $z < -\sqrt{2}$  as it respects the boundary conditions

Setting the boundary to the left of  $z = -\sqrt{2}$  does not affect the Coulomb gas - the wall does not touch the Wigner semi-circle

$$\Rightarrow S(z) = S(-\sqrt{2}) = S(-\infty) \text{ for } z < -\sqrt{2}$$

# Modified density function

QuickTime™ and a  
TIFF (Uncompressed) decompressor  
are needed to see this picture.

$z=1/2$

$z=-1$

$z=0$

Effect of moving the barrier position  $z$  on the eigenvalue  
density distribution

# Comparison with Numerics

QuickTime™ and a  
TIFF (Uncompressed) decompressor  
are needed to see this picture.

Analytic formula for  
f for  $z=0$  compared  
to that found by  
numerical diagonalizing  
of 6x6 matrices



# Computing $S(z)$

$$\frac{\mu^2}{2} + C = \int_z^\infty d\mu' f(\mu') \ln(|\mu - \mu'|) \Rightarrow$$

$$\int d\mu f(\mu) \frac{\mu^2}{4} + \frac{C}{2} = \frac{1}{2} \int d\mu d\mu' f(\mu) f(\mu') \ln(|\mu - \mu'|)$$

$$\Sigma[f] = \frac{1}{2} \int f(\mu) \mu^2 d\mu - \frac{1}{2} \int f(\mu) f(\mu') \ln(|\mu - \mu'|)$$

# Form of $S(z)$

QuickTime™ and a  
TIFF (Uncompressed) decompressor  
are needed to see this picture.

$$P_N(\lambda_{\min} > \zeta) = P_N(\mu_{\min} > z) = \frac{Z_N(z)}{Z_N(-\infty)} \quad ; z$$

# Matching with Tracy Widom

For

$$P(\lambda_{\min} > t) \approx \exp(-N^2 \beta \theta (\frac{t}{\sqrt{N}} + \cdot 2$$

# The persistence problem

Probability that all eigenvalues are positive  $\approx \exp(-N^2 \beta \theta(0))$

$$\theta(0) = \frac{\ln(3)}{4} \approx 0.274653..$$

Note that

$$0.25 < 0.274653.. < 0.3291$$

Aazami Easther  
aproximation

Exact result

Aazami Easther  
numerics

# Numerical test of persistence

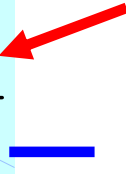
$$-0.272N^2 - 0.493N + 0.244$$

$\ln(P_N)$

QuickTime™ and a  
TIFF (Uncompressed) decompressor  
are needed to see this picture.

# Direct enumeraton

Can generate the ensembles and do the diagonalization efficiently but for large N you never see matrices with all eigenvalues positive

$$P(\lambda_{\min} > 0) = \frac{m_+}{m}$$


Trick of limited usefulness: if M is a positive matrix then:

$$(v, Mv) > 0 \quad \forall v \Rightarrow (e_i, Me_i) = M_{ii} > 0$$



# Conclusions

Coulomb gas formulation of eigenvalues of GOE  
GUE and GSE eigenvalues allows one to extract  
probability distribution of smallest eigenvalue far  
away from its typical value but within the sea

See a modified density of states in the conditioned  
ensemble quite rich behaviour

Is it possible to calculate the  $O(N)$  term and other  
lower order terms ?

Analysis of non symmetric matrix ensembles,  
Lyapounov exponents of dynamical fixed points  
of complex systems.

Are there more efficient ways of doing numerics ?

Can try to look at index distribution of random matrices